On the extended resolvent of the Nonstationary Schrödinger operator for a Darboux transformed potential

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Abstract. In the framework of the resolvent approach it is introduced a so called twisting operator that is able, at the same time, to superimpose à la Darboux N solitons to a generic smooth decaying potential of the Nonstationary Schrödinger operator and to generate the corresponding Jost solutions. This twisting operator is also used to construct an explicit bilinear representation in terms of the Jost solutions of the related extended resolvent. The main properties of the Jost and auxiliary Jost solutions and of the resolvent are discussed.

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1. Introduction

The Kadomtsev–Petviashvili equation in its version called KPI [1]–[3]

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}, \tag{1}$$

is a (2+1)-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation. As a consequence, the KPI equation admits solutions that behave at space infinity like the solutions of the KdV equation. For instance, if $u_1(t, x_1)$ obeys KdV, then $u(t, x_1, x_2) = u_1(t, x_1 + \mu x_2 + 3\mu^2 t)$ solves KPI for an arbitrary constant $\mu \in \mathbb{R}$. Thus, it is natural to consider solutions of (1) that are not decaying in all directions at space infinity but have 1-dimensional rays with behaviour of the type of u_1. Even though KPI has been known to be integrable for about three decades [2, 3], its general theory is far from being complete. Indeed, the Cauchy problem for KPI with rapidly decaying initial data was resolved in [4]–[9] by using the Inverse Scattering Transform (IST) method on the base of the spectral analysis of the Nonstationary Schrödinger operator

$$\mathcal{L}(x, i\partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - u(x), \qquad x = (x_1, x_2), \tag{2}$$

that gives the associated linear problem for the KPI equation. However, it is known that the standard approach to the spectral theory of the operator (2), based on integral equations for the Jost solutions, fails for potentials with one-dimensional asymptotic behaviour.

In [10]-[15] the method of the "extended resolvent" (or, for short, method of resolvent) was suggested as a way of pursuing a generalization of the IST that enables studying the spectral theory of operators with nontrivial asymptotic behaviour at space infinity. In [16]–[19] for the Nonstationary Schrödinger and heat operators the case where there is only one direction of nondecaying behaviour was considered. The starting point in solving the problem was the embedding of the pure one-dimensional case in the two-dimensional spectral theory, building the two-dimensional extended resolvent for a potential $u(x) \equiv u_1(x_1)$. Then, a potential $u(x) = u_1(x_1) + u_2(x)$, where $u_2(x)$ is an arbitrary decaying smooth function of both spatial variables, was considered and the corresponding resolvent was constructed by dressing the above resolvent for $u(x) = u_1(x_1)$. Finally, all mathematical entities generalizing the standard ones in IST, as Jost solutions and spectral data, were derived by a reduction procedure from this dressed resolvent.

Here, we consider the case of a potential not decaying along multiple non parallel rays, which is substantially more complicated since we have not the one-dimensional sample as a guide to follow and we must construct directly the resolvent, without passing trough the embedding of one-dimensional entities in two dimensions.

Therefore, we are obliged to consider directly true bidimensional potentials. In [20], by using recursively a binary Darboux transformation [21], it was constructed explicitly not only a two dimensional potential $\widetilde{u}$ which describes N solitons [22] of the most general form [23] "superimposed" to a generic background but also its Jost solutions. However, this recursive procedure seems not to be easily generalizable to the construction of the corresponding extended resolvent. On the other side, it is known (see [24]) that, in the framework of the extended resolvent approach, the whole hierarchy of time evolution equations related to $\mathcal{L}$ in (2), which can be considered as infinitesimal Darboux transformations, can be obtained by considering the similarity transformation $\widetilde{L} = \zeta L \zeta^\dagger$ of the extended version L of $\mathcal{L}$, where ζ is a convenient unitary operator. Here, we show that, by using a twisted transformation $\widetilde{L}\zeta = L\zeta^\dagger$

with ζ satisfying weaker conditions, one can bypass the recursive procedure and build directly the final potential $\widetilde{u}$ and Jost solutions. Then, we use this operator ζ, that we call twisting operator, to build directly the extended resolvent of $\widetilde{L}$ as a bilinear form in terms of the Jost solutions. The main properties of the Jost and so called auxiliary Jost solutions and of the resolvent are studied.

In a forthcoming paper, following the method developed in [15]–[19], we generalize the result obtained in this paper. Precisely, we consider the potential obtained by adding to the potential describing N solitons an arbitrary bidimensional smooth perturbation, we construct the corresponding extended resolvent by dressing this one obtained in this paper and we derive the corresponding Inverse Scattering problem.

2. Background theory

2.1. Extended operators and resolvent

In this section we briefly review the basic elements of the extended resolvent approach. For further details, we refer the interested readers to [10]–[16].

Let us consider the operators with kernel $L(x,x') = \mathcal{L}(x, i\partial_x)\delta(x-x')$ where $\mathcal{L}(x, i\partial_x)$ denotes a differential operator whose coefficients are smooth functions of x and let us introduce what we call the **extension** of these differential operators, i.e., to any differential operator $\mathcal{L}$ we associate the operator $L(\mathbf{q})$ with kernel

$$L(x,x';\mathbf{q}) \equiv \mathcal{L}(x, i\partial_x + \mathbf{q})\delta(x-x') = e^{i\mathbf{q}(x-x')}L(x,x'), \tag{3}$$

where $x = (x_1, x_2)$, $x' = (x'_1, x'_2) \in \mathbb{R}^2$, and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{C}^2$ and

$$\mathbf{q}x = \mathbf{q}_1 x_1 + \mathbf{q}_2 x_2.$$

The $\mathbf{q}$ variable will play in the following the role of a spectral parameter and we use a bold face character to emphasize that it is complex. By using the Fourier transform we can write

$$L(x,x';\mathbf{q}) = \frac{1}{(2\pi)^2}\int d\alpha\, e^{-i\alpha(x-x')}\mathcal{L}(x, \alpha+\mathbf{q}), \quad \alpha = (\alpha_1, \alpha_2).$$

Then, it is natural to introduce more general operators $A(\mathbf{q})$ with kernel

$$A(x,x';\mathbf{q}) = \frac{1}{(2\pi)^2}\int d\alpha\, e^{-i\alpha(x-x')}\mathcal{P}(x, \alpha+\mathbf{q}) \tag{4}$$

obtained by considering not just a polynomial $\mathcal{L}(x,\mathbf{q})$ in $\mathbf{q}$ but a tempered distribution $\mathcal{P}(x,\mathbf{q})$ of the six real variables x, $\mathbf{q}_\Re$ and $\mathbf{q}_\Im$. Notice that

$$A(x,x';\mathbf{q}) = e^{i\mathbf{q}_\Re(x-x')}A(x,x';q), \qquad q \equiv \mathbf{q}_\Im, \tag{5}$$

and that $A(x,x';q)$ belong to the space $\mathcal{S}'$ of tempered distributions of the six real variables x, x' and $q = (q_1, q_2)$. Since definition (4), up to the introduction of the spectral parameter $\mathbf{q}$ by shifting α, coincides with the definition of a pseudo-differential operator we shall call the operators belonging to the space $\mathcal{S}'$ extended pseudo-differential operators, or by short operators, and $\mathcal{P}(x,\mathbf{q})$ their symbol.

In the following it results often useful to use instead of the symbol $\mathcal{P}(x,\mathbf{q})$ its Fourier transform with respect to x, i.e.

$$A(p;\mathbf{q}) = \frac{1}{(2\pi)^2}\int dx e^{ipx}\mathcal{P}(x,\mathbf{q}), \qquad p = (p_1, p_2).$$

From (4) and (5) it follows that $A(p;\mathbf{q})$ is related to $A(x,x';q)$ by

$$A(p;\mathbf{q}) = \frac{1}{(2\pi)^2}\int dx \int dx' e^{i(p+\mathbf{q}_\Re)x - i\mathbf{q}_\Re x'} A(x,x';q). \tag{6}$$

Then, we consider $A(x,x';q)$ and $A(p;\mathbf{q})$ as the representation of the operator $A(q)$, respectively, in the x-space and in the p-space. The inverse of (6) is given by

$$A(x,x';\mathbf{q}_\Im) = \frac{1}{(2\pi)^2}\int dp \int d\mathbf{q}_\Re e^{-i(p+\mathbf{q}_\Re)x + i\mathbf{q}_\Re x'} A(p;\mathbf{q}). \tag{7}$$

On the space of this operators we define the hermitian conjugation as

$$A^\dagger(x,x';q) = \overline{A(x',x;-q)}, \qquad A^\dagger(p;\mathbf{q}) = \overline{A(-p;\overline{\mathbf{q}}+p)}, \tag{8}$$

in terms of kernels in x or p-spaces. For generic operators $A(q)$ and $B(q)$ with kernels $A(x,x';q)$ and $B(x,x';q)$ we introduce the standard composition law

$$(AB)(x,x';q) = \int dx''\, A(x,x'';q)\, B(x'',x';q), \tag{9}$$

if the integral exists in terms of distributions. In terms of kernels $A(p;\mathbf{q})$ and $B(p;\mathbf{q})$ this composition takes the form of a shifted convolution

$$(AB)(p;\mathbf{q}) = \int dp' A(p-p';\mathbf{q}+p')B(p';\mathbf{q}). \tag{10}$$

An operator A can have an inverse A^{-1} in the sense of this composition, i.e., such that $AA^{-1} = I$ or $A^{-1}A = I$, where I is the unity operator in $\mathcal{S}'$, $I(x,x';q) = \delta(x-x')$, being $\delta(x) = \delta(x_1)\delta(x_2)$ the two-dimensional δ-function (or $I(p;\mathbf{q}) = \delta(p)$ in p-space).

Of course the two representations in the x and in the p-space are equivalent and, in principle, one could work always in one of them. However, it results often convenient to pass from one representation to the other. Thus, the p-space is more suitable to study analyticity properties, while boundedness is more easily studied in the x-space.

The extension of the Nonstationary Schrödinger operator (2) is given by

$$L = L_0 - U \tag{11}$$

where in the x-space

$$L_0(x,x';q) = \left[i(\partial_{x_2}+q_2) + (\partial_{x_1}+q_1)^2\right]\delta(x-x'), \quad U(x,x';q) = u(x)\delta(x-x'). \tag{12}$$

The main object of our approach is the extended resolvent (or resolvent for short) $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator L, i.e.,

$$LM = ML = I. \tag{13}$$

Here we omit to specify the additional conditions that guarantee uniqueness of the extended resolvent as solution of (13), referring, say, to [18, 19].

For a real potential $u(x)$, as we always consider in the following, we have

$$L^\dagger = L, \qquad M^\dagger = M. \tag{14}$$

Now, let us consider in this section the case of a rapidly decaying potential $u(x)$ in (12).

One of the main advantage of the resolvent approach is that the dressing operators can be obtained directly from the resolvent by means of a truncation and reduction procedure. Thus, here the dressing operators ν and ω are defined by

$$\nu(p;\mathbf{q}_1) = (ML_0)(p;\mathbf{q})\Big|_{\mathbf{q}=\ell(\mathbf{q}_1)}, \qquad \omega(p;\mathbf{q}_1) = (L_0M)(p;\mathbf{q})\Big|_{\mathbf{q}=\ell(\mathbf{q}_1+p_1)-p}, \tag{15}$$

where we introduced the special two-component vector

$$\ell(\mathbf{k}) = (\mathbf{k}, \mathbf{k}^2). \tag{16}$$

In fact these operators dress, in the sense proposed and developed by Zakharov–Shabat [2], the operator $L(q)$ and its resolvent $M(q)$. More precisely, they admit the following bilinear representation in terms of ν and ω

$$L = \nu L_0 \omega, \qquad M = \nu M_0 \omega, \tag{17}$$

where L_0 and M_0 are the bare operators

$$L_0(p; \mathbf{q}) = \delta(p)(\mathbf{q}_2 - \mathbf{q}_1^2), \qquad M_0(p; \mathbf{q}) = \frac{\delta(p)}{\mathbf{q}_2 - \mathbf{q}_1^2}. \tag{18}$$

Dressing operators are mutually adjoint

$$\nu^\dagger = \omega, \tag{19}$$

mutually inverse

$$\omega\nu = I, \tag{20}$$

$$\nu\omega = I, \tag{21}$$

and obey the equations

$$L\nu = \nu L_0, \qquad \omega L = L_0 \omega. \tag{22}$$

The kernels of these operators in p-space obey asymptotic

$$\lim_{\mathbf{q}_1 \to \infty} \nu(p; \mathbf{q}) = \delta(p), \qquad \lim_{\mathbf{q}_1 \to \infty} \omega(p; \mathbf{q}) = \delta(p), \tag{23}$$

are independent of $\mathbf{q}_2$ and analytic functions of the variable $\mathbf{q}_1$ in the upper and lower half planes.

Spectral data and the inverse problem can be formulated in this operatorial approach. Here, we give not details but only the main formulae, that will be useful in the following. Let $\nu^\pm$ and $\omega^\pm$ denote the operators with kernels being the limiting values of $\nu(p; \mathbf{q})$ and $\omega(p; \mathbf{q})$ at the real $\mathbf{q}_1$-axis from above and below

$$\nu^\pm(p; \mathbf{q}_1) = \nu(p; \mathbf{q}_{1\Re} \pm i0), \qquad \omega^\pm(p; \mathbf{q}_1) = \omega(p; \mathbf{q}_{1\Re} \pm i0). \tag{24}$$

Then we get the relations

$$\nu^\pm = \nu^\mp F^\mp, \qquad \omega^\pm = F^\pm \omega^\mp, \tag{25}$$

where we introduced the spectral data

$$F^\pm = \omega^\pm \nu^\mp. \tag{26}$$

By construction the kernels $F^\pm(p; \mathbf{q})$ depend on three real variables p_1, p_2, and $\mathbf{q}_{1\Re}$ and thanks to (19)–(22) these operators obey

$$L_0(q) F^\pm = F^\pm L_0(q), \quad \text{when} \quad q_1 = 0, \tag{27}$$

$$(F^\pm)^\dagger = F^\pm, \tag{28}$$

$$F^+ F^- = I. \tag{29}$$

By means of the last equality and (23) it can be shown that the kernels $F^\pm(p; \mathbf{q})$ have the representation

$$F^\pm(p; \mathbf{q}) = \delta(p) + \delta(p_2 - p_1(p_1 + 2\mathbf{q}_{1\Re})) f^\pm(p_1 + \mathbf{q}_{1\Re}, \mathbf{q}_{1\Re}), \tag{30}$$

where the functions $f^\pm(p_1, \mathbf{q}_{1\Re})$ depend only on two real variables.

2.2. Hat-kernels, Green's functions and Jost solutions

Let us associate to any operator $A(q)$ with kernel $A(x,x';q)$ its "hat-kernel"

$$\widehat{A}(x,x';q)=e^{q(x-x')}A(x,x';q). \tag{31}$$

If $A(q)$ is an extended differential operator $L(q)$ then this procedure is the inverse of (3), i.e., $\widehat{L}(x,x';q)=\mathcal{L}(x,x')$, while for a generic operator the hat-kernel can continue to depend on q. For any extended differential operator $L(q)$ and for any (not necessary differential) operator $B(q)$ the following relations hold

$$\begin{aligned}(\widehat{LB})(x,x';q)&=\mathcal{L}(x,\partial_x)\widehat{B}(x,x';q),\\ (\widehat{BL})(x,x';q)&=\mathcal{L}^{\mathrm{d}}(x',\partial_{x'})\widehat{B}(x,x';q),\end{aligned}$$

where $\mathcal{L}^{\mathrm{d}}$ is the operator dual to $\mathcal{L}$. In particular, by (13) we have that

$$\mathcal{L}(x,\partial_x)\widehat{M}(x,x';q)=\mathcal{L}^{\mathrm{d}}(x',\partial_{x'})\widehat{M}(x,x';q)=\delta(x-x'),$$

so that the hat-kernel of the resolvent defines a family of Green's function depending on the bidimensional parameter q. In particular, the Jost solutions are defined by means of the following Green's function depending on a complex spectral parameter $\mathbf{k}$

$$\mathcal{G}(x,x',\mathbf{k})=\widehat{M}(x,x';\ell_{\Im}(\mathbf{k})), \tag{32}$$

with the vector $\ell(\mathbf{k})$ defined as in (16), while the advanced/retarded solutions are defined by the Green's functions $\mathcal{G}_{\pm}(x,x')$ obtained by the following limiting procedure

$$\mathcal{G}_{\pm}(x,x')=\lim_{q_2\to\pm0}\lim_{q_1\to0}M(x,x';q). \tag{33}$$

The Jost solution $\Phi(x,\mathbf{k})$ and its dual $\Psi(x,\mathbf{k})$ are defined by

$$\Phi(x,\mathbf{k})=\int dx' e^{-i\ell(\mathbf{k})x'}\mathcal{L}_0^{\mathrm{d}}(x',\partial_{x'})\mathcal{G}(x,x',\mathbf{k}) \tag{34}$$

$$\Psi(x',\mathbf{k})=\int dx\, e^{i\ell(\mathbf{k})x}\mathcal{L}_0(x,\partial_x)\mathcal{G}(x,x',\mathbf{k}), \tag{35}$$

or by using definitions (15) in terms of the dressing operators by

$$\Phi(x,\mathbf{k})=e^{-i\ell(\mathbf{k})x}\chi(x,\mathbf{k}),\qquad \Psi(x,\mathbf{k})=e^{i\ell(\mathbf{k})x}\xi(x,\mathbf{k}), \tag{36}$$

where

$$\chi(x,\mathbf{q}_1)=\int dp\, e^{-ipx}\nu(p;\mathbf{q}_1),\qquad \xi(x,\mathbf{q}_1)=\int dp\, e^{-ipx}\omega(p;\mathbf{q}_1-p_1). \tag{37}$$

The $\mathbf{q}_2$ independence of the kernels $\nu(p;\mathbf{q})$ and $\omega(p;\mathbf{q})$ implies, thanks to (7) and (31), that the corresponding hat-kernels $\nu(x,x';q)$ and $\omega(x,x';q)$ are independent of q_2 and are proportional to $\delta(x_2-x_2')$. Thus, instead of (36) and (37) we get relations

$$\Phi(x,\mathbf{q}_1)=\int dy\,\widehat{\nu}(x,y;\mathbf{q}_{1\Im})e^{-i\ell(\mathbf{q}_1)y},\qquad \Psi(x,\mathbf{q}_1)=\int dy\, e^{i\ell(\mathbf{q}_1)y}\widehat{\omega}(y,x;\mathbf{q}_{1\Im}), \tag{38}$$

Writing (13) in the form $M=M_0+M_0UM$ one can derive from (34) and (35) the standard integral equation for the Jost solution in the case of a smooth potential rapidly decaying at space infinity

$$\Phi(x,\mathbf{k})=e^{-i\ell(\mathbf{k})x}+\int dx'\mathcal{G}_0(x,x',\mathbf{k})u(x')\Phi(x',\mathbf{k}), \tag{39}$$

where

$$\mathcal{G}_0(x,x',\mathbf{k}) = \frac{\operatorname{sgn} x_2}{2\pi i}\int d\alpha\, \theta(\alpha\mathbf{k}_{\Im}x_2)\, e^{i\ell(\alpha-\mathbf{k})(x-x')},$$

is the Green's function, introduced in [5]. In our approach this Green's function follows directly from the second equality in (18), transformation (7), and definition (32). Solvability of this integral equation under some small norm assumptions was proved in [8] and thanks to (39) it is easy to show that $\chi(x,\mathbf{k})$ has the asymptotic behaviour

$$\lim_{x\to\infty}\chi(x,\mathbf{k}) = 1. \tag{40}$$

Properties of the dressing operators lead to corresponding properties of the Jost solutions. Thus (19) gives

$$\overline{\Phi(x,\mathbf{k})} = \Psi(x,\overline{\mathbf{k}}), \qquad \overline{\chi(x,\mathbf{k})} = \xi(x,\overline{\mathbf{k}}), \quad \mathbf{k}\in\mathbb{C}. \tag{41}$$

Property (23) is equivalent to

$$\lim_{\mathbf{k}\to\infty}\chi(x,\mathbf{k}) = 1, \qquad \lim_{\mathbf{k}\to\infty}\xi(x,\mathbf{k}) = 1, \tag{42}$$

relations (20) and (21) can be considered as scalar product and completeness of the Jost solutions:

$$\frac{1}{2\pi}\int dx_1 \Phi(x,\mathbf{k})\Psi(x,\mathbf{k}+\alpha) = \delta(\alpha), \quad \mathbf{k}\in\mathbb{C}, \quad \alpha\in\mathbb{R}, \tag{43}$$

$$\frac{1}{2\pi}\int d\mathbf{k}_{\Re}\Phi(x,\mathbf{k})\Psi(y,\mathbf{k})\Big|_{x_2=y_2} = \delta(x_1-y_1); \tag{44}$$

and equalities (22) are equivalent to the Nonstationary Schrödinger equation and its dual:

$$\mathcal{L}(x,\partial_x)\Phi(x,\mathbf{k}) = 0, \qquad \mathcal{L}^{\mathrm{d}}(x,\partial_x)\Psi(x,\mathbf{k}) = 0. \tag{45}$$

The Jost solutions are analytic in the complex plane of the spectral parameter $\mathbf{k}$, $\mathbf{k}_{\Im}\neq 0$. Using for the boundary values at the real $\mathbf{k}$-axis notations of the type (24) we get from (25) and (30) the standard [5] nonlocal Riemann–Hilbert problem

$$\Phi^{\pm}(x,k) = \Phi^{\mp}(x,k) + \int d\alpha\, \Phi^{\mp}(x,\alpha) f^{\mp}(\alpha,k), \quad k\in\mathbb{R}. \tag{46}$$

3. Twisting transformation

Here, we construct a two dimensional potential together with its Jost solutions which describes N solitons [22, 23] "superimposed" to a generic background by using a twisted transformation $\widetilde{L}\zeta = \zeta L$ from the extended differential operator L in (11) to a new operator $\widetilde{L}$ of the same form obtained by means of an operator ζ which is isometric but not necessarily unitary. The use of the twisting operator ζ allows us, bypassing the usual procedure consisting in applying recursively binary Darboux transformations, to get directly not only the Jost and auxiliary Jost solutions but also the extended resolvent, which results to be a bilinear expression in terms of these Jost and auxiliary Jost solutions.

3.1. Properties of twisting operator ζ

Let us consider twisting the operator L in (11) to a new operator $\widetilde{L}$ of the same kind

$$\widetilde{L} = L_0 - \widetilde{U}, \qquad \widetilde{U}(x, x'; q) = \widetilde{u}(x)\delta(x - x'), \tag{47}$$

by means of an operator ζ according to the formula

$$\widetilde{L}\zeta = \zeta L. \tag{48}$$

We consider a potential $u(x)$ in L which is real, smooth and rapidly decaying at space infinity and we search for a ζ such that the new potential $\widetilde{u}(x)$ is also real and smooth, while condition of rapid decaying is not imposed. Notice that since L and $\widetilde{L}$ are both selfadjoint from (48) it follows that

$$\zeta^\dagger \widetilde{L} = L\zeta^\dagger. \tag{49}$$

In addition we require that ζ obeys the conditions

(**I**) the operator ζ is isometric

$$\zeta^\dagger \zeta = I, \tag{50}$$

but not necessarily unitary;

(**II**) the kernel $\zeta(p; \mathbf{q})$ is independent of $\mathbf{q}_2$,

$$\zeta(p; \mathbf{q}) = \zeta(p; \mathbf{q}_1); \tag{51}$$

(**III**) the kernel $\zeta(p; \mathbf{q})$ obeys the asymptotic condition

$$\lim_{\mathbf{q}_1 \to \infty} \zeta(p; \mathbf{q}) = \delta(p). \tag{52}$$

Then, a specific transformation ζ is chosen by fixing its analyticity properties in $\mathbf{q}_1$.

Notice that because ζ is not unitary the operator

$$P = I - \zeta\zeta^\dagger, \tag{53}$$

is not zero and since it satisfies

$$P^\dagger = P \tag{54}$$

and, thanks to (50),

$$P^2 = P, \tag{55}$$

it is an orthogonal projector.

We fix the analyticity properties of ζ by requiring that it generates not only the new potential $\widetilde{u}$ via (48) but also the new dressing operators $\widetilde{\nu}$ and $\widetilde{\omega}$. In fact, taking into account (22) we get by (48) and (49)

$$\widetilde{L}\zeta\nu = \zeta\nu L_0, \qquad \omega\zeta^\dagger \widetilde{L} = L_0 \omega\zeta^\dagger,$$

and, therefore, the operators $\widetilde{\nu}$ and $\widetilde{\omega}$ defined by

$$\widetilde{\nu}\tau = \zeta\nu, \tag{56}$$

$$\tau^\dagger \widetilde{\omega} = \omega\zeta^\dagger, \tag{57}$$

for any operator τ commuting with L_0 satisfy the equations

$$\widetilde{L}\widetilde{\nu} = \widetilde{\nu} L_0, \qquad \widetilde{\omega}\widetilde{L} = L_0 \widetilde{\omega}. \tag{58}$$

analogous to the equations (22) satisfied by dressing operators ν and ω of L. Since $\widetilde{\nu}$ and $\widetilde{\omega}$ are are mutually adjoint

$$\widetilde{\nu}^{\dagger} = \widetilde{\omega}, \tag{59}$$

we can limit ourself to consider $\widetilde{\nu}$ and say that $\widetilde{\nu}$ can be considered the new dressing operator of $\widetilde{L}$ if the operator τ and consequently ζ is chosen in such a way that the kernel of the $\widetilde{\nu}$ in p-space

(**IV**) independent of $\mathbf{q}_2$

$$\widetilde{\nu}(p;\mathbf{q}) = \widetilde{\nu}(p;\mathbf{q}_1), \tag{60}$$

(**V**) is an analytic function of the variable $\mathbf{q}_1$ in the upper and lower half planes, continuous on the two sides of the real axis and obeys the asymptotic behaviour

$$\lim_{\mathbf{q}_1\to\infty} \widetilde{\nu}(p;\mathbf{q}) = \delta(p). \tag{61}$$

Since ν, ω, and ζ by (**II**), are independent of $\mathbf{q}_2$ we deduce from (**IV**) that also τ must be independent of $\mathbf{q}_2$ and, then, since τ commutes with L_0, that its kernel has the form

$$\tau(p;\mathbf{q}) = \delta(p)\tau(\mathbf{q}_1). \tag{62}$$

Moreover, from (**V**) we get that $\tau(\mathbf{q}_1)$ has asymptotic behaviour

$$\lim_{\mathbf{q}_1\to\infty} \tau(\mathbf{q}_1) = 1. \tag{63}$$

Taking into account (20) and (50) we get from (56) and (57) the scalar product of the new dressing operators

$$\widetilde{\omega}\widetilde{\nu} = T^{-1}, \tag{64}$$

where we introduced the selfadjoint operator

$$T = \tau\tau^{\dagger} \tag{65}$$

with kernel

$$T(p;\mathbf{q}) = \delta(p)t(\mathbf{q}_1) \tag{66}$$

where thanks to the composition law (10) and definition (8)

$$t(\mathbf{q}_1) = \tau(\mathbf{q}_1)\overline{\tau(\overline{\mathbf{q}}_1)}. \tag{67}$$

From (64), thanks to (**V**), we deduce that $1/t(\mathbf{q}_1)$ and therefore $1/\tau(\mathbf{q}_1)$ must be chosen to be a function analytic in the upper and lower half $\mathbf{q}_1$-planes and continuous on the two sides of the real axis. Therefore, we assume that

(**VI**) $\tau(\mathbf{q}_1)$ is meromorphic in the upper and lower half planes without zeros and with a finite number of poles and satisfies asymptotic (63).

Consequently, $t(\mathbf{q}_1)$, which plays the role of transmission coefficient, is analytic in the upper and lower half $\mathbf{q}_1$-planes, continuous on the two sides of the real axis with no zeros and with poles at the poles of $\tau(\mathbf{q}_1)$ and $\overline{\tau(\overline{\mathbf{q}}_1)}$.

The completeness relation of the new dressing operators, which will play a crucial role in the following, can be written in terms of T and the projection operator P. In fact from (56), (57), and (53) we get

$$\widetilde{\nu}T\widetilde{\omega} + P = I. \tag{68}$$

It is also worth to mention that the operator P annihilates the new dressing operators. Indeed, from (64) and (68) we have

$$P\widetilde{\nu} = \widetilde{\omega}P = 0. \tag{69}$$

Finally, by using the completeness relation (21) from (56) we have

$$\zeta = \widetilde{\nu}\tau\omega \tag{70}$$

and, once given an operator τ satisfying the above described conditions, the analyticity properties of ζ are fixed and consequently ζ itself. However, in determining the analyticity properties of ζ we have not yet required that also the new dressing operators $\widetilde{\nu}$ and $\widetilde{\omega}$ satisfy non local Riemann-Hilbert problems analogous to them satisfied by ν and ω in (25). This will be done in the next section.

3.2. Transformed continuous spectrum

In order to describe the discontinuity at the real axis in the complex $\mathbf{q}_1$ plane of the twisting operator ζ and of the new dressing operators and Jost solutions we use notations (24) and mention that for any operator A by definition (8) we have

$$(A^{\pm})^{\dagger} = (A^{\dagger})^{\mp}. \tag{71}$$

From (56) in the limits $q_1 \to \pm 0$ we get

$$\widetilde{\nu}^{\pm}\tau^{\pm} = \zeta^{\pm}\nu^{\pm}. \tag{72}$$

Multiplying from the left by $(\zeta^{\dagger})^{\pm}$, thanks to (50), we get

$$\nu^{\pm} = (\zeta^{\dagger})^{\pm}\widetilde{\nu}^{\pm}\tau^{\pm} \tag{73}$$

Inserting (25) into the r.h.s. of (72) and then inserting $\nu^{\mp}$ from (73) in the obtained equation we get

$$\widetilde{\nu}^{\pm}\tau^{\pm} = \zeta^{\pm}(\zeta^{\dagger})^{\mp}\widetilde{\nu}^{\mp}\tau^{\mp}F^{\mp}.$$

If we impose that $\widetilde{\nu}$ satisfies a Riemann-Hilbert problem analogous to that one in (25) satisfied by ν we deduce, taking into account that according to (69) $\widetilde{\nu}^{\mp}$ has a left annihilator $P^{\mp}$, that

$$\zeta^{\pm}(\zeta^{\dagger})^{\mp} = c_1 I + c_2(I + P^{\mp}) = c_1 I + c_2\zeta^{\mp}(\zeta^{\dagger})^{\mp}$$

with c_1 and c_2 constant. Then, since ζ satisfies the asymptotic property (52) thanks to (50) we derive

$$\zeta^{+} = \zeta^{-}, \tag{74}$$

that is

(VII) the twisting operator ζ is continuous on the real axis of the $\mathbf{q}_1$ plane.

Let us, now, introduce the operator

$$\widetilde{F}^{\pm} = \tau^{\pm}F^{\pm}(\tau^{\pm})^{\dagger}. \tag{75}$$

Its kernel equals (see (30))

$$\widetilde{F}^{\pm}(p;\mathbf{q}) = \delta(p)|\tau^{\pm}(\mathbf{q}_{1\Re})|^2 + \delta(p_2 - p_1(p_1 + 2\mathbf{q}_{1\Re}))\widetilde{f}^{\pm}(p_1 + \mathbf{q}_{1\Re}, \mathbf{q}_{1\Re}), \tag{76}$$

where

$$\widetilde{f}^{\pm}(p_1, \mathbf{q}_{1\Re}) = \tau^{\pm}(p_1)f^{\pm}(p_1, \mathbf{q}_{1\Re})\overline{\tau^{\pm}}(\mathbf{q}_{1\Re}). \tag{77}$$

Thanks to properties of $\tau^{\pm}$, the operators $\widetilde{F}^{\pm}$ have properties (27) and (28). Taking into account (71) and (66) we get finally

$$\widetilde{\nu}^{\pm}T^{\pm}=\widetilde{\nu}^{\mp}\widetilde{F}^{\mp},\qquad T^{\pm}\widetilde{\omega}^{\pm}=\widetilde{F}^{\pm}\widetilde{\omega}^{\mp}, \tag{78}$$

where the second equation is obtained by conjugation. Comparison of (78) with (25) motivates the definition of $\widetilde{F}^{\pm}$ as the continuous spectrum of the new potential $\widetilde{u}$. Again, thanks to (71) and (66), we see that relations (29) and (26) are modified, respectively, as

$$(T^{\pm})^{-1}\widetilde{F}^{\pm}(T^{\mp})^{-1}\widetilde{F}^{\mp}=I, \tag{79}$$

and as

$$\widetilde{\omega}^{\pm}\widetilde{\nu}^{\mp}=(T^{\pm})^{-1}\widetilde{F}^{\pm}(T^{\mp})^{-1}. \tag{80}$$

In the case where the background potential $u(x)$ is identically equal to zero the second term in (30) is absent and $F^{\pm}=I$. It is natural to preserve this property for the new spectral data, so we impose

(**VIII**) unitarity:

$$|\tau^{\pm}(\mathbf{q}_{1\Re})|=1, \tag{81}$$

that in terms of operators means that

$$(\tau^{\pm})^{\dagger}=\left(\tau^{\pm}\right)^{-1}.$$

In analogy with (36) and (37) we define

$$\widetilde{\chi}(x,\mathbf{q}_1)=\int dp\, e^{-ipx}\widetilde{\nu}(p;\mathbf{q}_1),\qquad \widetilde{\xi}(x,\mathbf{q}_1)=\int dp\, e^{-ipx}\widetilde{\omega}(p;\mathbf{q}_1-p_1), \tag{82}$$

$$\widetilde{\Phi}(x,\mathbf{k})=e^{-i\ell(\mathbf{k})x}\widetilde{\chi}(x,\mathbf{k}),\qquad \widetilde{\Psi}(x,\mathbf{k})=e^{i\ell(\mathbf{k})x}\widetilde{\xi}(x,\mathbf{k}), \tag{83}$$

where notation (16) is used. These functions can be defined also by relations generalizing (38). Indeed, taking into account that the function $\tau(\mathbf{q}_1)$ has no zeroes we have by (56) and (57) that $\widetilde{\nu}(p;\mathbf{q}_1)=(\zeta\nu)(p;\mathbf{q})/\tau(\mathbf{q}_1)$ and $\widetilde{\omega}(p;\mathbf{q}_1)=(\omega\zeta^{\dagger})(p;\mathbf{q})/\overline{\tau(p_1+\overline{\mathbf{q}}_1)}$. Inserting this equalities in (82) we derive by (6), (31), and (38) that

$$\widetilde{\Phi}(x,\mathbf{q}_1)=\frac{1}{\tau(\mathbf{q}_1)}\int dy\,\widehat{\zeta}(x,y;\mathbf{q}_{1\Im})\Phi(y,\mathbf{q}_1), \tag{84}$$

$$\widetilde{\Psi}(x,\mathbf{q}_1)=\frac{1}{\overline{\tau(\overline{\mathbf{q}}_1)}}\int dy\,\Psi(y,\mathbf{q}_1)\widehat{\zeta}^{\dagger}(y,x;\mathbf{q}_{1\Im}). \tag{85}$$

In what follows we prove that $\widetilde{\Phi}$ is the Jost solution of operator $\widetilde{\mathcal{L}}$ and $\widetilde{\Psi}$ is the Jost solution of the dual operator. Thanks to (59) we have like in (41)

$$\overline{\widetilde{\Phi}(x,\mathbf{k})}=\widetilde{\Psi}(x,\overline{\mathbf{k}}),\qquad \overline{\widetilde{\chi}(x,\mathbf{k})}=\widetilde{\xi}(x,\overline{\mathbf{k}}),\quad \mathbf{k}\in\mathbb{C}. \tag{86}$$

Under condition (**V**) functions $\widetilde{\Phi}(x,\mathbf{k})$, $\widetilde{\Psi}(x,\mathbf{k})$, $\widetilde{\chi}(x,\mathbf{k})$, and $\widetilde{\xi}(x,\mathbf{k})$ are analytic with respect to the variable $\mathbf{k}\in\mathbb{C}$ with only discontinuity at the real axis and functions $\widetilde{\chi}(x,\mathbf{k})$, $\widetilde{\xi}(x,\mathbf{k})$ obey asymptotic condition (42). Eq.(78) enables us to write down relations between boundary values $\widetilde{\Phi}^{\pm}(x,k)$ and $\widetilde{\Psi}^{\pm}(x,k)$ of these functions on the real axis. Indeed, taking into account definition (10) of composition we get by (66), (76), and (81)

$$\widetilde{\Phi}^{\pm}(x,k)t^{\pm}(k)=\widetilde{\Phi}^{\mp}(x,k)+\int d\alpha\,\widetilde{\Phi}^{\mp}(x,\alpha)\widetilde{f}^{\mp}(\alpha,k),\quad k\in\mathbb{R}, \tag{87}$$

that modifies relation (46), valid in the case of rapidly decaying potential $u(x)$.

3.3. Twisting transformation of the resolvent

The operator ζ, once obtained the transformed operator $\widetilde{L}$, can also be used to get the corresponding resolvent $\widetilde{M} = \widetilde{L}^{-1}$. From (48) and the definition of the resolvent in (13) we have that $\widetilde{M}$ and M are related by the same relation as $\widetilde{L}$ and L, i.e.,

$$\widetilde{M}\zeta = \zeta M. \tag{88}$$

Multiplying (48) and (88) from the right by $\zeta^{\dagger}$ and recalling the definition (53) of P we get

$$\widetilde{L} = \zeta L \zeta^{\dagger} + L_{\Delta}, \qquad \widetilde{M} = \zeta M \zeta^{\dagger} + M_{\Delta} \tag{89}$$

with

$$L_{\Delta} = \widetilde{L}P = P\widetilde{L}, \qquad M_{\Delta} = \widetilde{M}P = P\widetilde{M}, \tag{90}$$

where the second terms in both equations follow by hermitian conjugation. Since P is a projection operator we get directly

$$M_{\Delta}P = PM_{\Delta} = M_{\Delta}, \tag{91}$$

$$L_{\Delta}M_{\Delta} = M_{\Delta}L_{\Delta} = P. \tag{92}$$

Viceversa, for M_{Δ} satisfying (91) and (92) the operator $\widetilde{M}$ in (89) is the resolvent of $\widetilde{L}$. In fact we have by (50), (13), and (92)

$$\widetilde{L}\widetilde{M} = (\zeta L \zeta^{\dagger} + L_{\Delta})(\zeta M \zeta^{\dagger} + M_{\Delta}) = I + \zeta L \zeta^{\dagger} M_{\Delta} + L_{\Delta}\zeta M \zeta^{\dagger}. \tag{93}$$

To prove that the last two terms are zero we, first, note that thanks to (50) and (53) we have that

$$P\zeta = \zeta^{\dagger}P = 0,$$

and, then, that by (90) $\zeta^{\dagger}M_{\Delta} = \zeta^{\dagger}P\widetilde{M} = 0$ and as well $L_{\Delta}\zeta = \widetilde{L}P\zeta = 0$. So

$$\widetilde{L}\widetilde{M} = \widetilde{M}\widetilde{L} = I, \tag{94}$$

where the second equation is obtained by conjugation.

In conclusion, in order to obtain M_{Δ}, and consequently $\widetilde{M}$ via (89), one can, first, find the general solution X (belonging to our space of operators) of the system

$$PX = X, \qquad XP = X, \tag{95}$$

and, then, M_{Δ} will be the special X satisfying

$$L_{\Delta}X = XL_{\Delta} = P. \tag{96}$$

3.4. Construction of operator ζ

In correspondence with condition (**VI**) let $\tau(\mathbf{q}_1)$ have poles at $\mathbf{q}_1 = \overline{\lambda}_1, \dots, \overline{\lambda}_N$ and let us suppose, for simplicity, that they are simple and that

$$\lambda_m \neq \lambda_n, \quad \lambda_m \neq \overline{\lambda_n}, \quad \forall\, m \neq n$$
$$\lambda_{n\Im} \neq 0, \quad \forall\, n. \tag{97}$$

From properties (**VI**) and (**VIII**) of τ by dispersion relation we get that

$$\tau(\mathbf{q}_1) = \prod_{n=1}^{N} \left(\frac{\mathbf{q}_1 - \lambda_n}{\mathbf{q}_1 - \overline{\lambda}_n} \right)^{\theta(-\mathbf{q}_{1\Im}\lambda_{n\Im})} \equiv \prod_{n=1}^{N} \frac{\mathbf{q}_1 - \lambda_{n\Re} + i|\lambda_{n\Im}| \operatorname{sgn} \mathbf{q}_{1\Im}}{\mathbf{q}_1 - \overline{\lambda}_n}, \tag{98}$$

For the residua of this function we have

$$\tau_m = \operatorname*{res}_{\mathbf{q}_1=\overline{\lambda}_m} \tau(\mathbf{q}_1) \equiv -2i\lambda_{m\Im} \prod_{n=1,\, n\neq m}^{N} \left(\frac{\overline{\lambda}_m - \lambda_n}{\overline{\lambda}_n - \overline{\lambda}_m}\right)^{\theta(\lambda_{m\Im}\lambda_{n\Im})} . \tag{99}$$

From (67) we get for the transmission coefficient

$$t(\mathbf{q}_1) = \prod_{n=1}^{N} \left(\frac{\mathbf{q}_1 - \overline{\lambda}_n}{\mathbf{q}_1 - \lambda_n}\right)^{\operatorname{sgn}(\mathbf{q}_{1\Im}\lambda_{n\Im})} \equiv \prod_{n=1}^{N} \left(\frac{\mathbf{q}_1 - \lambda_{n\Re} + i|\lambda_{n\Im}|}{\mathbf{q}_1 - \lambda_{n\Re} - i|\lambda_{n\Im}|}\right)^{\operatorname{sgn}(\mathbf{q}_{1\Im})} . \tag{100}$$

The residua at the poles at $\mathbf{q}_1 = \lambda_m$ and $\mathbf{q}_1 = \overline{\lambda}_m$ are given by

$$t_m = \operatorname*{res}_{\mathbf{q}_1=\lambda_m} t(\mathbf{q}_1) \equiv 2i\lambda_{m\Im} \prod_{n=1,\, n\neq m}^{N} \left(\frac{\lambda_m - \overline{\lambda}_n}{\lambda_m - \lambda_n}\right)^{\operatorname{sgn}(\lambda_{m\Im}\lambda_{n\Im})} , \tag{101}$$

$$\operatorname*{res}_{\mathbf{q}_1=\overline{\lambda}_m} t(\mathbf{q}_1) = \overline{t}_m. \tag{102}$$

An alternative expression is given by

$$t_m = \tau(\lambda_m)\overline{\tau}_m. \tag{103}$$

Now, we have all we need for building the twisting operator ζ. First, we use (21) to rewrite (56) as

$$\zeta = \widetilde{\nu}\tau\omega. \tag{104}$$

Now, from the analyticity properties in $\mathbf{q}_1$ of $\widetilde{\nu}$ stated in (**IV**), (**V**), of τ as given in (98) and of ω, from the continuity of ζ on the real $\mathbf{q}_1$-axis stated in (**VII**) and the asymptotic property of $\zeta(p;\mathbf{q}_1)$ we have that $\zeta(p;\mathbf{q})$ satisfies the following integral representation

$$\zeta(p;\mathbf{q}) = \delta(p) + \sum_{n=1}^{N} \tau_n \int dp' \, \frac{\widetilde{\nu}(p-p';\overline{\lambda}_n)\omega(p';\overline{\lambda}_n - p'_1)}{\mathbf{q}_1 + p'_1 - \overline{\lambda}_n}. \tag{105}$$

The kernel $\zeta(p;\mathbf{q})$ results to be piecewise analytic with discontinuities along the lines $\mathbf{q}_{1\Im} = -\lambda_{n\Im}$ and it results do depend only on the values of $\widetilde{\nu}(p;\mathbf{q}_1)$ at $\mathbf{q}_1 = \overline{\lambda}_n$. Therefore, in order to get ζ it is sufficient to construct the new dressing operator $\widetilde{\nu}$ at the special values of poles of τ.

The kernel of ζ in the x-space is given by (6). Then, for its hat-kernel (see (31)) we get

$$\begin{aligned}\widehat{\zeta}(x,x';q) &= \delta(x-x') - i\operatorname{sgn}(x_1-x'_1)\delta(x_2-x'_2) \\ &\quad\times \sum_{n=1}^{N} \tau_n \theta((q_1+\lambda_{n\Im})(x_1-x'_1))\widetilde{\Phi}(x,\overline{\lambda}_n)\Psi(x',\overline{\lambda}_n),\end{aligned} \tag{106}$$

where we used notations (36), (37), (82), and (83). By conjugation (see (8), (41), and (86)) we derive

$$\begin{aligned}\widehat{\zeta}^{\dagger}(x,x';q) &= \delta(x-x') - i\operatorname{sgn}(x_1-x'_1)\delta(x_2-x'_2) \\ &\quad\times \sum_{n=1}^{N} \overline{\tau}_n \theta((q_1-\lambda_{n\Im})(x_1-x'_1))\Phi(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n).\end{aligned} \tag{107}$$

In order to complete the construction of ζ we have to impose the isometry condition (50). Using (106) and (107) we get that this condition is equivalent to

$$\delta(x_2-x_2')\sum_{n=1}^{N}\overline{\tau}_n\theta((q_1-\lambda_{n\Im})(x_1-x_1'))\Phi(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n)$$
$$+\,\delta(x_2-x_2')\sum_{n=1}^{N}\tau_n\theta((q_1+\lambda_{n\Im})(x_1-x_1'))\widetilde{\Phi}(x,\overline{\lambda}_n)\Psi(x',\overline{\lambda}_n)$$
$$=i\,\mathrm{sgn}(x_1-x_1')\delta(x_2-x_2')\sum_{m,n=1}^{N}\overline{\tau}_m\tau_n\Phi(x,\lambda_m)\Psi(x',\overline{\lambda}_n)\,\mathrm{sgn}(q_1-\lambda_{m\Im})\,\mathrm{sgn}(q_1+\lambda_{n\Im})$$
$$\times\int dy_1\theta((q_1-\lambda_{m\Im})(x_1-y_1))\theta((q_1+\lambda_{n\Im})(y_1-x_1'))\widetilde{\Psi}(y,\lambda_m)\widetilde{\Phi}(y,\overline{\lambda}_n)\Big|_{y_2=x_2}.$$

Let us, now, introduce the matrix

$$\Theta(x)=\left\|\overline{\tau}_m\tau_n\int\limits_{x_1}^{(\lambda_m+\lambda_n)_\Im\infty}dy_1\widetilde{\Psi}(y,\lambda_m)\widetilde{\Phi}(y,\overline{\lambda}_n)\Big|_{y_2=x_2}\right\|_{m,n=1}^{N},\tag{108}$$

where the factor $(\lambda_m+\lambda_n)_\Im$ in the limit of integration defines the sign of the infinity. Thanks to (83) it is easy to check that these integrals are well defined if $\widetilde{\xi}(x,\lambda_m)$ and $\widetilde{\chi}(x,\overline{\lambda}_n)$ are bounded at space infinity, and that by (86) this matrix is hermitian:

$$\overline{\Theta_{mn}(x)}=\Theta_{nm}(x).\tag{109}$$

Thus

$$\overline{\tau}_m\tau_n\widetilde{\Psi}(x,\lambda_m)\widetilde{\Phi}(x,\overline{\lambda}_n)=-\partial_{x_1}\Theta_{mn}(x),\tag{110}$$

and the above condition is simplified to

$$\sum_{m=1}^{N}\theta((q_1-\lambda_{m\Im})(x_1-x_1'))\Phi(x,\lambda_m)\left[\overline{\tau}_m\widetilde{\Psi}(x',\lambda_m)-i\sum_{n=1}^{N}\Theta_{mn}(x')\Psi(x',\overline{\lambda}_n)\right]$$
$$+\sum_{m=1}^{N}\theta((q_1+\lambda_{m\Im})(x_1-x_1'))\left[\tau_m\widetilde{\Phi}(x,\overline{\lambda}_m)+i\sum_{n=1}^{N}\Phi(x,\lambda_n)\Theta_{nm}(x)\right]\Psi(x',\overline{\lambda}_m)=0,$$

where $x_2=x_2'$. The function in the l.h.s. is piecewise constant with respect to q_1, so this condition is equivalent to the equalities

$$\widetilde{\Phi}(x,\overline{\lambda}_m)=\frac{1}{i\tau_m}\sum_{n=1}^{N}\Phi(x,\lambda_n)\Theta_{nm}(x),\tag{111}$$

$$\widetilde{\Psi}(x,\lambda_m)=\frac{i}{\overline{\tau}_m}\sum_{n=1}^{N}\Theta_{mn}(x)\Psi(x,\overline{\lambda}_n).\tag{112}$$

By (106) in order to construct the hat kernel of the operator ζ we have to determine the functions $\widetilde{\Phi}(\overline{\lambda}_m)$ and $\widetilde{\Psi}(\lambda_m)$. Thanks to (111) and (112) this means that we have to determine the matrix Θ_{mn}. For this sake we insert these equations in (110), that gives

$$\partial_{x_1}\Theta_{mn}(x)=-\sum_{m',n'=1}^{N}\Theta_{mm'}(x)\Psi(x,\overline{\lambda}_{m'})\Phi(x,\lambda_{n'})\Theta_{n'n}(x),$$

that under assumption of invertibility of the matrix $\Theta(x)$ can be rewritten as

$$\partial_{x_1}\Theta^{-1}_{mn}(x) = \Psi(x,\overline{\lambda}_m)\Phi(x,\lambda_n). \tag{113}$$

Let us introduce in analogy with (108) the matrix

$$B_{mn}(x) = \int\limits_{-(\lambda_m+\lambda_n)_{\Im}\infty}^{x_1} dy_1\Psi(y,\overline{\lambda}_m)\Phi(y,\lambda_n)\Big|_{y_2=x_2}, \tag{114}$$

where again the limits of integration are uniquely determined by the asymptotic behaviour of $\Phi(x,\overline{\lambda}_m)$ and $\Psi(x,\lambda_n)$ given by (36) and (40). Moreover,

$$\lim_{x\to\infty} B_{mn}(x)e^{-i(\overline{\ell(\lambda_m)}-\ell(\lambda_n))x} = \frac{-i}{\overline{\lambda}_m-\lambda_n},$$

and by (41) this matrix is hermitian,

$$\overline{B_{mn}(x)} = B_{nm}(x).$$

Now (113) gives

$$\Theta_{mn}(x) = (B(x)+C)^{-1}_{mn}, \tag{115}$$

where we introduced the matrix $C = \|c_{mn}\|_{m,n=1}^{N}$ with matrix elements independent of x_1.

Let $C_\pm$ denote the matrices constructed from C as

$$C_\pm = \|c_{mn};\, m,n=1,\ldots,N;\, \pm\lambda_{m\Im}>0,\, \pm\lambda_{n\Im}>0\|, \tag{116}$$

and $C_+ = I$ ($C_- = I$) if all $\lambda_{n\Im}$ are negative (positive). In [20] it was shown that if the matrix C is hermitian and obeys the positiveness condition

$$\pm\, C_\pm > 0 \tag{117}$$

the determinant $\det(B(x)+C)$ has no zeros on the x-plane including infinity (for the case of zero background potential this result was obtained in [23]). Thus, the matrix $\Theta(x)$ given by (115) exists for any x and the r.h.s.'s of (111) and (112) are given explicitly in terms of the Jost solutions of the background potential $u(x)$. Such matrix $\Theta(x)$ is hermitian indeed, so that by (41) equalities (111) and (112) give a special case of (86) for $\mathbf{k}=\overline{\lambda}_m$.

Finally, inserting (111) in (106) and (112) in (107) we get the hat-kernels of ζ, $\zeta^\dagger$. Multiplying them by $e^{-q(x-x')}$ (see (31)) we get kernels

$$\zeta(x,x';q) = \delta(x-x') - \operatorname{sgn}(x_1-x_1')\delta(x_2-x_2')e^{-q_1(x_1-x_1')} \times \sum_{m,n=1}^{N}\theta((q_1+\lambda_{n\Im})(x_1-x_1'))\Phi(x,\lambda_m)\Theta_{mn}(x)\Psi(x',\overline{\lambda}_n), \tag{118}$$

$$\zeta^\dagger(x,x';q) = \delta(x-x') + \operatorname{sgn}(x_1-x_1')\delta(x_2-x_2')e^{-q_1(x_1-x_1')} \times \sum_{m,n=1}^{N}\theta((q_1-\lambda_{m\Im})(x_1-x_1'))\Phi(x,\lambda_m)\Theta_{mn}(x')\Psi(x',\overline{\lambda}_n), \tag{119}$$

proving by (41) and (109) that they are mutually conjugate in the sense of (8).

3.5. Transformed dressing operators and Jost solutions

By using the constructed twisting operator we can obtain the dressing operators by (56) and (57), while expressions for functions introduced in (83) follow from (84) and (85). Indeed, inserting (118) and (119) in these relations, we get

$$\widetilde{\Phi}(x,\mathbf{k}) = \frac{1}{\tau(\mathbf{k})}\left[\Phi(x,\mathbf{k}) - \sum_{m,n=1}^{N}\Phi(x,\lambda_m)\Theta_{mn}(x)\beta_n(x,\mathbf{k})\right], \tag{120}$$

$$\widetilde{\Psi}(x,\mathbf{k}) = \frac{1}{\overline{\tau(\overline{\mathbf{k}})}}\left[\Psi(x,\mathbf{k}) - \sum_{m,n=1}^{N}\overline{\beta_m(x,\overline{\mathbf{k}})}\Theta_{mn}(x)\Psi(x,\overline{\lambda}_n)\right], \tag{121}$$

where we denoted

$$\beta_n(x,\mathbf{k}) = \int\limits_{-(\mathbf{k}+\lambda_n)_{\Im}\infty}^{x_1} dy_1\Psi(y,\overline{\lambda}_n)\Phi(y,\mathbf{k})\Big|_{y_2=x_2}, \tag{122}$$

so that by (114)

$$B_{mn}(x) = \beta_m(x,\lambda_n) = \overline{\beta_n(x,\lambda_m)}.$$

Thanks to the properties of the Jost solutions Φ and Ψ, the functions $\beta_n(\mathbf{k})$ are analytic in the complex domain of $\mathbf{k}$ with exception of the real axis and of a pole at $\mathbf{k} = \overline{\lambda}_n$, where by (43)

$$\operatorname*{res}_{\mathbf{k}=\overline{\lambda}_n} \beta_n(x,\mathbf{k}) = i.$$

By (99) this proves that $\widetilde{\Phi}(x,\overline{\lambda}_m)$ ($\widetilde{\Psi}(x,\lambda_m)$) given in (111) (correspondingly, (112)) are values at $\mathbf{k} = \overline{\lambda}_m$ (at $\mathbf{k} = \lambda_m$) of (120) (correspondingly, (121)).

Taking into account the well known property of the determinants of bordered matrices

$$\frac{1}{\det\Gamma_n}\begin{vmatrix}\Gamma_n & \Gamma_{*,n+1}\\ \Gamma_{n+1,*} & \gamma_{n+1,n+1}\end{vmatrix} = \gamma_{n+1,n+1} - \Gamma_{n+1,*}\Gamma_n^{-1}\Gamma_{*,n+1},$$

it is easy to see that (120) is exactly the Jost solution constructed in [20] as result of N successive "binary" Bäcklund transformations. It is annulated by the operator

$$\widetilde{\mathcal{L}}(x,i\partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - \widetilde{u}(x), \tag{123}$$

iff the matrix C in (115) is independent also on x_2. Correspondingly, $\widetilde{\Psi}(x,\mathbf{k})$ is the Jost solution of the dual operator. In (123) the new potential (cf. (2)) is given by means of the well known [21] relation

$$\widetilde{u}(x) = u(x) + 2\partial_{x_1}^2 \log\det\Theta(x). \tag{124}$$

We proved in [20] that this potential is smooth, real and finite for all x under condition (117) and that it decays in all directions on the x-plane with exception of a finite number of directions $x_1 - 2\lambda_{j\Re}x_2 =$const, where it tends to a unidimensional soliton. It was also proved there that $\widetilde{\chi}(x,\mathbf{k})$ (see (83)) is a bounded function of its variables, analytic in $\mathbf{k}$ with a discontinuity at the real axis, and obeying asymptotic condition (42), while instead of (40) we have

$$\lim_{x_1\to-\mathbf{k}_{\Im}\infty}\widetilde{\chi}(x,\mathbf{k}) = 1, \qquad \lim_{x_1\to\mathbf{k}_{\Im}\infty}\widetilde{\chi}(x,\mathbf{k}) = \frac{1}{t(\mathbf{k})}. \tag{125}$$

The Jost solutions given by (120) and (121) obviously obey (86). Then properties of $\widetilde{\xi}(x,\mathbf{k})$ are the same up to the asymptotic

$$\lim_{x_1\to-\mathbf{k}_\Im\infty}\widetilde{\xi}(x,\mathbf{k})=\frac{1}{t(\mathbf{k})},\qquad \lim_{x_1\to\mathbf{k}_\Im\infty}\widetilde{\xi}(x,\mathbf{k})=1, \tag{126}$$

where we used that by (67) $\overline{t(\overline{\mathbf{k}})}=t(\mathbf{k})$.

Summarizing, we have that kernels $\zeta(x,x';q)$, $\zeta^\dagger(x,x';q)$ and $\widetilde{\nu}(p;\mathbf{q})$, $\widetilde{\omega}(p;\mathbf{q})$ (as given by (82)) belong to the space $\mathcal{S}'(\mathbb{R}^6)$, so they define operators in the sense of definitions given in Sec. 2. Moreover, these operators obey all conditions (**II**)–(**V**) that were imposed. Asymptotic behaviour (125) and (126) shows that the values of the Jost solutions at the poles of $t(\mathbf{k})$, like in the one-dimensional case, have special relevance. Thus in addition to the values given in (111) we have to consider $\Phi(x,\lambda_j)$. Thanks to (122) and (115) we get from (120)

$$\widetilde{\Phi}(x,\lambda_j)=\frac{1}{\tau(\lambda_j)}\sum_{m,n=1}^{N}\Phi(x,\lambda_m)\Theta_{mn}(x)c_{mj}.$$

then by (100), (103), and (111), (112)

$$\widetilde{\Phi}(x,\lambda_m)=\frac{i}{t_m}\sum_{n=1}^{N}\widetilde{\Phi}(x,\overline{\lambda}_n)\tau_n c_{nm}\overline{\tau}_m, \tag{127}$$

$$\widetilde{\Psi}(x,\overline{\lambda}_m)=\frac{1}{i\overline{t}_m}\sum_{n=1}^{N}\tau_m c_{mn}\overline{\tau}_n\widetilde{\Psi}(x,\lambda_n). \tag{128}$$

Using these equalities we get

$$\sum_{n=1}^{N}\Big\{t_n\widetilde{\Phi}(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n)+\overline{t}_n\widetilde{\Phi}(x,\overline{\lambda}_n)\widetilde{\Psi}(x',\overline{\lambda}_n)\Big\}=0. \tag{129}$$

Relation (127) together with (87) and the first equality in (42) close the formulation of the inverse problem for the Jost solution $\widetilde{\Phi}(x,\mathbf{k})$. Analogously, (128), (87) and the second equality in (42) give the inverse problem for the dual Jost solution $\widetilde{\Psi}(x,\mathbf{k})$.

3.6. Boundedness of $\widetilde{\Phi}(x,\lambda_n)$, $\widetilde{\Phi}(x,\overline{\lambda}_n)$ *and their dual*

Here we prove that the Jost solutions $\widetilde{\Phi}(x,\lambda_n)$, $\widetilde{\Phi}(x,\overline{\lambda}_n)$ and $\widetilde{\Psi}(x,\lambda_n)$, $\widetilde{\Psi}(x,\overline{\lambda}_n)$ are bounded when x tends to infinity, while specific asymptotic behaviour essentially depends on the direction of the limiting procedure. Let us notice that $\widetilde{\Phi}(x,\lambda_n)$ and $\widetilde{\Phi}(x,\overline{\lambda}_n)$ can be written as

$$\widetilde{\Phi}(x,\lambda_n)=e^{\ell_\Im(\lambda_n)x-i\ell_\Re(\lambda_n)x}\widetilde{\chi}(x,\lambda_n),\quad \widetilde{\Phi}(x,\overline{\lambda}_n)=e^{-\ell_\Im(\lambda_n)x+i\ell_\Re(\lambda_n)x}\widetilde{\chi}(x,\overline{\lambda}_n),$$

where $\widetilde{\chi}(x,\lambda_n)$ and $\widetilde{\chi}(x,\overline{\lambda}_n)$ are known to be bounded. In the limit $x\to\infty$ along a direction of the x-plane some of the exponents $e^{\ell_\Im(\lambda_n)x}$ are increasing, some are decreasing or bounded. Taking into account that new potential, Jost solutions, and spectral data by construction are symmetric functions of $\lambda_1,\ldots,\lambda_N$, we renumber them in a way that, say $e^{\ell_\Im(\lambda_n)x}$ for $n=1,\ldots,s-1$ are decreasing or constant and $e^{\ell_\Im(\lambda_n)x}$ for $n=s,\ldots,N$ are increasing, where $s=1,\ldots,N+1$ is a number depending on the direction on the x-plane. Then $\widetilde{\Phi}(x,\lambda_n)$ are decreasing for $n=s,\ldots,N$ and

$\widetilde{\Phi}(x,\overline{\lambda}_n)$ are bounded when $n=1,\ldots,s-1$. In order to consider the complementary intervals we write (127) for $m\geq s$ in the form

$$\sum_{n=s}^{N}\widetilde{\Phi}(x,\overline{\lambda}_n)\tau_n c_{nm}=\frac{t_m}{i\overline{\tau}_m}\widetilde{\Phi}(x,\lambda_m)-\sum_{n=1}^{s-1}\widetilde{\Phi}(x,\overline{\lambda}_n)\tau_n c_{nm}.$$

Thanks to conditions (117) matrix C cannot have zero eigenvalue. Then the same is valid for the matrix $\|c_{mn}\|_{m,n=s}^{N}$ in the l.h.s. of this equality. Taking into account that all terms in the r.h.s. are bounded, we conclude that $\widetilde{\Phi}(x,\overline{\lambda}_n)$ are bounded also in the interval $n=s,\ldots,N$ and then for all n. Now by (127) the same is valid for $\widetilde{\Phi}(x,\lambda_n)$. Boundedness of $\widetilde{\Psi}(x,\lambda_n)$, $\widetilde{\Psi}(x,\overline{\lambda}_n)$ follows by conjugation.

4. Transformed resolvent

4.1. Operator P and completeness relation

We already mentioned in Sec. 3.1 the essential role played by the operator P. Here we derive an explicit expression for its kernel and present its properties. Inserting (106) and (107) in (53) and using (114) we get

$$\begin{aligned}\widehat{P}(x,x';q)&=i\,\mathrm{sgn}(x_1-x_1')\delta(x_2-x_2')\sum_{m=1}^{N}\tau_m\theta((q_1+\lambda_{m\Im})(x_1-x_1'))\\&\quad\times\widetilde{\Phi}(x,\overline{\lambda}_m)\left[\Psi(x',\overline{\lambda}_m)+i\sum_{n=1}^{N}\overline{\tau}_n\widetilde{\Psi}(x',\lambda_n)B_{mn}(x')\right]\\&\quad+i\,\mathrm{sgn}(x_1-x_1')\delta(x_2-x_2')\sum_{n=1}^{N}\overline{\tau}_n\theta((q_1-\lambda_{n\Im})(x_1-x_1'))\\&\quad\times\left[\Phi(x,\lambda_n)-i\sum_{m=1}^{N}\tau_m\widetilde{\Phi}(x,\overline{\lambda}_m)B_{mn}(x)\right]\widetilde{\Psi}(x',\lambda_n)\end{aligned}$$

By (111) $\Phi(x,\lambda_n)=i\sum_{m=1}^{N}\tau_m\widetilde{\Phi}(x,\overline{\lambda}_m)(\ \Theta(x))_{mn}^{-1}$, so thanks to (115) and (127)

$$\Phi(x,\lambda_n)-i\sum_{m=1}^{N}\tau_m\widetilde{\Phi}(x,\overline{\lambda}_m)B_{mn}(x)=\frac{t_n}{\overline{\tau}_n}\widetilde{\Phi}(x,\lambda_n).$$

Thanks to this equality and its complex conjugate we get

$$\begin{aligned}\widehat{P}(x,x';q)&=i\delta(x_2-x_2')\,\mathrm{sgn}(x_1-x_1')\\&\quad\times\sum_{n=1}^{N}\Big[t_n\theta((q_1-\lambda_{n\Im})(x_1-x_1'))\widetilde{\Phi}(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n)\\&\quad+\overline{t}_n\theta((q_1+\lambda_{n\Im})(x_1-x_1'))\widetilde{\Phi}(x,\overline{\lambda}_n)\widetilde{\Psi}(x',\overline{\lambda}_n)\Big].\end{aligned}$$

The form of this expression shows a discontinuity of the r.h.s. at $x_1=x_1'$, while thanks to (129) the actual discontinuity is absent. In order to exploit this fact directly we write

$$\begin{aligned}\widehat{P}(x,x';q)&=i\delta(x_2-x_2')\theta(q_1)\\&\quad\times\sum_{n=1}^{N}\Big\{t_n\Big[\theta(\lambda_{n\Im})\theta(|q_1|-|\lambda_{n\Im}|)\theta(x_1-x_1')+\theta(-\lambda_{n\Im})\theta(x_1-x_1')\end{aligned}$$

$$- \theta(\lambda_{n\Im})\theta(|\lambda_{n\Im}| - |q_1|)\theta(x_1' - x_1)\Big]\widetilde{\Phi}(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n)$$
$$+ \overline{t}_n\Big[\theta(\lambda_{n\Im})\theta(x_1 - x_1') + \theta(-\lambda_{n\Im})\theta(|q_1| - |\lambda_{n\Im}|)\theta(x_1 - x_1')$$
$$- \theta(-\lambda_{n\Im})\theta(|\lambda_{n\Im}| - |q_1|)\theta(x_1' - x_1)\Big]\widetilde{\Phi}(x,\overline{\lambda}_n)\widetilde{\Psi}(x',\overline{\lambda}_n)\Big\} + \text{h.c.},$$

where hermitian conjugation is understood in the sense of (8), so that

$$\begin{aligned}\widehat{P}(x,x';q) = {} & i\delta(x_2 - x_2')\operatorname{sgn} q_1 \theta(q_1(x_1 - x_1')) \\ & \times \sum_{n=1}^{N}\Big\{t_n\widetilde{\Phi}(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n) + \overline{t}_n\widetilde{\Phi}(x,\overline{\lambda}_n)\widetilde{\Psi}(x',\overline{\lambda}_n)\Big\} \\ & - i\delta(x_2 - x_2')\operatorname{sgn} q_1 \sum_{n=1}^{N}\theta(|\lambda_{n\Im}| - |q_1|) \\ & \times\Big\{t_n\theta(q_1\lambda_{n\Im})\widetilde{\Phi}(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n) + \overline{t}_n\theta(-q_1\lambda_{n\Im})\widetilde{\Phi}(x,\overline{\lambda}_n)\widetilde{\Psi}(x',\overline{\lambda}_n)\Big\}.\end{aligned}$$

Again, thanks to (129) the first term cancels out and we get

$$\begin{aligned}\widehat{P}(x,x';q) = {} & -i\delta(x_2 - x_2')\operatorname{sgn} q_1 \sum_{n=1}^{N}\theta(|\lambda_{n\Im}| - |q_1|) \\ & \times\Big\{t_n\theta(q_1\lambda_{n\Im})\widetilde{\Phi}(x,\lambda_n)\widetilde{\Psi}(x',\lambda_n) + \overline{t}_n\theta(-q_1\lambda_{n\Im})\widetilde{\Phi}(x,\overline{\lambda}_n)\widetilde{\Psi}(x',\overline{\lambda}_n)\Big\}. \quad (130)\end{aligned}$$

This suggests the introduction of q_1-dependent solutions of the Nonstationary Schrödinger equation

$$\widetilde{\Phi}_n(x,q_1) = \theta(|\lambda_{n\Im}| - |q_1|)\Big\{\theta(q_1\lambda_{n\Im})\widetilde{\Phi}(x,\lambda_n) + \theta(-q_1\lambda_{n\Im})\widetilde{\Phi}(x,\overline{\lambda}_n)\Big\}, \quad (131)$$

$$\widetilde{\Psi}_n(x,q_1) = \theta(|\lambda_{n\Im}| - |q_1|)\Big\{\theta(q_1\lambda_{n\Im})\widetilde{\Psi}(x,\lambda_n) + \theta(-q_1\lambda_{n\Im})\widetilde{\Psi}(x,\overline{\lambda}_n)\Big\}, \quad (132)$$

which we call auxiliary Jost solutions.

Then we rewrite (130) as

$$\widehat{P}(x,x';q) = -i\operatorname{sgn} q_1 \delta(x_2 - x_2')\sum_{n=1}^{N}\vartheta_n(q_1)\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1) \quad (133)$$

where

$$\vartheta_n(q_1) = \theta(q_1\lambda_{n\Im})t_n + \theta(-q_1\lambda_{n\Im})\overline{t}_n. \quad (134)$$

Thus the operator P results to be a sum of operators that are different from zero in the intervals $[-|\lambda_{n\Im}|, |\lambda_{n\Im}|]$ and continuous at $x_1 = x_1'$. In particular,

$$P(q) = 0 \quad \text{for} \quad |q_1| > \max_n |\lambda_{n\Im}|.$$

Finally, inserting into (68) equation (66) and equations (82), (83) relating the dressing operators to the Jost solutions and the expression obtained for P in (133) we get the completeness relation for the Jost solutions

$$\begin{aligned}&\frac{1}{2\pi}\int d\alpha\,\widetilde{\Phi}(x;\alpha + iq_1)t(\alpha + iq_1)\widetilde{\Psi}(x';\alpha + iq_1) \\ & \quad - i\operatorname{sgn} q_1 \sum_{n=1}^{N}\vartheta_n(q_1)\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1) = \delta(x_1 - x_1'), \quad \text{for } x_2 = x_2'. \quad (135)\end{aligned}$$

4.2. Properties of auxiliary Jost solutions

Functions $\widetilde{\Phi}_n(x,q_1)$ and $\widetilde{\Psi}_n(x,q_1)$ by definitions (131), (132) obey the nonstationary Schrödinger equation and, correspondingly, its dual. Their conjugation property

$$\overline{\widetilde{\Phi}_n(x,q_1)} = \widetilde{\Psi}_n(x,-q_1),$$

follows from (86). These functions are different from zero on the interval $|q_1| < |\lambda_{n\Im}|$ only, that guaranties boundedness of $e^{-q_1x_1}\widetilde{\Phi}_n(x,q_1)$ and $e^{q_1x_1}\widetilde{\Psi}_n(x,q_1)$ when x_1 goes to infinity. As the result the kernel $P(x,x';q)$ defined by (31) is tempered distribution with respect to all its variables that proves that operator $P(q)$ belongs to the space of operators under consideration. The naturally appeared above piecewise dependence of functions $\widetilde{\Phi}_n(x,q_1)$ and $\widetilde{\Psi}_n(x,q_1)$ on q_1 remembers the same, but nontrivial, dependence discovered in studying the perturbation of the one-line potential in [18, 19]. It is necessary to mention that functions $\widetilde{\Phi}_n(x,q_1)$ and $\widetilde{\Psi}_n(x,q_1)$ are discontinuous at $q_1 = 0$, while thanks to (129) $P(q)$ is continuous at this point.

To study the discontinuity of these functions at $q_1 = 0$ we use the standard notations (cf. (24)) for the right and left limit at $q_1 = 0$ and we write

$$\widetilde{\Phi}^{\pm}_n(x) = \theta(\pm\lambda_{n\Im})\widetilde{\Phi}(x,\lambda_n) + \theta(\mp\lambda_{n\Im})\widetilde{\Phi}(x,\overline{\lambda}_n),$$
$$\widetilde{\Psi}^{\pm}_n(x) = \theta(\pm\lambda_{n\Im})\widetilde{\Psi}(x,\lambda_n) + \theta(\mp\lambda_{n\Im})\widetilde{\Psi}(x,\overline{\lambda}_n),.$$

To find relations between these limiting values we have to partially invert relations in (127) and (128). Taking into account that the whole our construction here is explicitly symmetric with respect to the λ_n's, we can renumber them in a way that the first ones for $n = 1,\ldots,s-1$ are in the upper half plane and the last ones for $n = s,\ldots,N$ are in the lower half plane. Correspondingly the matrix C decomposes as follows

$$C = \begin{pmatrix} C_+ & Z \\ Z^\dagger & C_- \end{pmatrix}, \tag{136}$$

where $C_\pm$ are the matrices, positive and negative definite, introduced in (117), and where Z, $Z^\dagger$, which are mutually adjoint, are rectangular matrices just filling the C matrix. By introducing the vectors

$$\Phi_+(\lambda) = (\widetilde{\Phi}^+_1,\ldots,\widetilde{\Phi}^+_{s-1}) \quad \Phi_-(\lambda) = (\widetilde{\Phi}^-_s,\ldots,\widetilde{\Phi}^-_N)$$
$$\Phi_+(\overline{\lambda}) = (\widetilde{\Phi}^-_1,\ldots,\widetilde{\Phi}^-_{s-1}) \quad \Phi_-(\overline{\lambda}) = (\widetilde{\Phi}^+_s,\ldots,\widetilde{\Phi}^+_N)$$

and the diagonal matrices

$$\tau = \mathrm{diag}\{\tau_j\}, \qquad t = \mathrm{diag}\{t_j\},$$

decomposed in the diagonal matrices $\tau_\pm$ and $t_\pm$ in analogy to (136), equation (127) can be rewritten as

$$\Phi_+(\lambda)t_+ = i\Phi_+(\overline{\lambda})\tau_+C_+\tau^\dagger_+ + i\Phi_-(\overline{\lambda})\tau_-Z^\dagger\tau^\dagger_+,$$
$$\Phi_-(\lambda)t_- = i\Phi_+(\overline{\lambda})\tau_+Z\tau^\dagger_- + i\Phi_-(\overline{\lambda})\tau_-C_-\tau^\dagger_-.$$

Solving the last equality for $\Phi_-(\overline{\lambda})$ and inserting the result into the first one, we get

$$\widetilde{\Phi}^+_m(x) = i\sum_{n=1}^{N}\widetilde{\Phi}^-_n(x)\vartheta^+_n c'_{nm},$$

where the matrix $C' = \|c'_{mn}\|_{m,n=1}^{N}$ equals

$$C' = \begin{pmatrix} (t_+^{-1})^\dagger\tau_+ & 0 \\ 0 & (\tau_-^{-1})^\dagger \end{pmatrix} \begin{pmatrix} C_+ - ZC_-^{-1}Z^\dagger & iZC_-^{-1} \\ -iC_-^{-1}Z^\dagger & -C_-^{-1} \end{pmatrix} \begin{pmatrix} \tau_+^\dagger t_+^{-1} & 0 \\ 0 & \tau_-^{-1} \end{pmatrix}$$

and ϑ_n^+ is one of the limiting values of (134):

$$\vartheta_n^\pm = \theta(\pm\lambda_{n\Im})t_n + \theta(\mp\lambda_{n\Im})\overline{t}_n,$$

that in analogy with (101), (102) can also be defined as

$$\vartheta_n^\pm = \operatorname*{res}_{\mathbf{k}=\lambda_{n\Re}\mp i|\lambda_{n\Im}|} t(\mathbf{k}).$$

Thanks to conditions on matrix C given in section 3.4, the matrix C' is hermitian and positive. The first property is obvious. In order to check the second one we write for an arbitrary vector (v_+, v_-)

$$(v_+^\dagger, v_-^\dagger) \begin{pmatrix} C_+ - ZC_-^{-1}Z^\dagger & iZC_-^{-1} \\ -iC_-^{-1}Z^\dagger & -C_-^{-1} \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$$
$$= v_+^\dagger C_+ v_+ - [v_- - iZ^\dagger v_+]^\dagger C_-^{-1}[v_- + iZ^\dagger v_+],$$

that proves that conditions (117) are equivalent to condition $C' > 0$.

Now relation for the functions $\widetilde{\Psi}_n^\pm(x)$ follows by complex conjugation, taking into account that

$$\overline{\widetilde{\Phi}_n^\pm(x)} = \widetilde{\Psi}_n^\mp(x), \qquad \overline{\vartheta_n^\pm} = \vartheta_n^\mp.$$

Let us mention that the only dependence on the signs of $\lambda_{n\Im}$ enters in the functions $\tau(\mathbf{k})$ in (98), and then in the spectral data (76). All other objects of our constructions depend on $|\lambda_{n\Im}|$ only. This means that in the case of the zero background potential $u(x)$, when both $f^\pm$ and $\widetilde{f}^\pm$ are identically equal to zero, without loss of generality we can chose in addition to (97) $\lambda_{n\Im} > 0$ $(n = 1, 2, \ldots, N)$.

On the other side in the case of the nonzero background potential solutions corresponding to λ_n with opposite sign are different, as was shown in [20].

Finally, let us mention that relation $P^2 = P$ gives the scalar product

$$\int dx_1\, \widetilde{\Psi}_m(x, q_1)\widetilde{\Phi}_n(x, q_1) = \frac{i\delta_{mn}\theta(|\lambda_{m\Im}| - |q_1|)}{(\operatorname{sgn} q_1)\vartheta_m(q_1)}, \tag{137}$$

that takes a more complicated form if we turn back to the original solutions.

4.3. Operators L_Δ and M_Δ

In order to find L_Δ we can use the first equality in (90) applying $\widetilde{L}$ to P. Then in terms of the hat-kernels we use that $\widetilde{\Phi}_m^\pm(x)$ is annulated by $\widetilde{\mathcal{L}}$, so

$$\widehat{L}_\Delta(x, x'; q) = \operatorname{sgn} q_1 \delta'(x_2 - x_2') \sum_{n=1}^{N} \vartheta_n(q_1)\widetilde{\Phi}_n(x, q_1)\widetilde{\Psi}_n(x', q_1). \tag{138}$$

Now it is easy to check directly that

$$PL_\Delta = L_\Delta P = L_\Delta,$$

as it also follows from from (90) and (55).

In Sec. 3.3 we proved that in order to obtain M_Δ it is enough to find, first, a general (belonging to our space of operators) solution X of the system (95) and then to find a special X satisfying (96).

The expression for $\widehat{L}_\Delta$ in (138) suggests that the general solution X of (95) is given by

$$\widehat{X}(x,x';q)=\sum_{n=1}^{N}\vartheta_n(q_1)g_m(x_2,x_2';q)\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1), \tag{139}$$

with $g_m(x_2,x_2';q)$ such that $X(x,x';q)\in\mathcal{S}'$, but otherwise arbitrary function of the written arguments. This can be verified directly by using the representation of $\widehat{P}$ and the orthogonality (137).

Thus we take $\widehat{M_\Delta}$ in the form (139) and, according to the discussion at the end of Sec. 3.1, we fix the unknown function by requiring that equations (96) for $X=M_\Delta$ are satisfied. We get

$$\widehat{M_\Delta}(x,x';q)=-\operatorname{sgn}q_1\sum_{n=1}^{N}\vartheta_n(q_1)\{\theta(x_2-x_2')+g_m(q)\}\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1), \tag{140}$$

with $g_m(q)$ to be chosen in such a way that $M_\Delta(x,x';q)$ is bounded at space infinity. It results that

$$\begin{aligned}M_\Delta(x,x';q)=&-\operatorname{sgn}q_1\operatorname{sgn}(x_2-x_2')e^{-q(x-x')}\\&\times\sum_{n=1}^{N}\vartheta_n(q_1)\theta((q_2-2\lambda_{n\Re}q_1)(x_2-x_2'))\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1)\end{aligned} \tag{141}$$

obtained for

$$g_m(q)=-\theta(-q_2+2\lambda_{m\Re}q_1) \tag{142}$$

is a well defined bounded operator.

Since the $\widetilde{\Phi}$'s and $\widetilde{\Psi}$'s are bounded we need only to consider the region of variables where the exponent $e^{-q(x-x')}$ is growing. Since, thanks to the choice (142) in each term in the r.h.s. of (140)

$$-q_2(x_2-x_2')\le-2\lambda_{n\Re}q_1(x_2-x_2')$$

the behaviour of $e^{-q(x-x')}$ for $q_1>0$ cannot be worse than $e^{-q_1((x_1-x_1')+2\lambda_{n\Re}(x_2-x_2'))}$ and, then, since $|q_1|\le|\lambda_{n\Im}|$, no worse than $\theta(\pm\lambda_{n\Im})e^{\mp\ell_\Im(\lambda_n)(x-x')}$. But in the interval $0\le q_1\le|\lambda_{n\Im}|$ where Φ_n and Ψ_n are different from zero

$$\begin{aligned}&\theta(\pm\lambda_{n\Im})e^{\mp\ell_\Im(\lambda_n)(x-x')}\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1)\\&\qquad=\theta(\pm\lambda_{n\Im})e^{-i\ell_\Re(\lambda_n)(x-x')}\widetilde{\chi}(x,\lambda_{n_\Re}+i|\lambda_{n\Im}|)\widetilde{\xi}(x',\lambda_{n_\Re}+i|\lambda_{n\Im}|)\end{aligned}$$

is bounded. Analogously for $q_1<0$. This proves that M_Δ is bounded.

We are left with the explicit construction of $\zeta M\zeta^\dagger$. Form (70) and the conjugation properties of the Jost solutions we have

$$\zeta M\zeta^\dagger=\widetilde{\nu}\tau\omega M\nu\tau^\dagger\widetilde{\omega}$$

and, then, from (17), (20) and (66)

$$\zeta M\zeta^\dagger=\widetilde{\nu}TM_0\widetilde{\omega}.$$

In the p-space this reads

$$(\zeta M\zeta^\dagger)(p;\mathbf{q})=\int dp'\widetilde{\nu}(p-p';\mathbf{q}_1+p_1')\frac{t(\mathbf{q}_1+p_1')}{\mathbf{q}_2+p_2'-(\mathbf{q}_1+p_1')^2}\widetilde{\omega}(p';\mathbf{q}_1)$$

and shifting p', that is naming $p' + \mathbf{q}_{\Re} = \alpha$,

$$(\zeta M \zeta^{\dagger})(p;\mathbf{q}) = \int d\alpha\, \widetilde{\nu}(p-\alpha+\mathbf{q}_{\Re};\alpha_1+iq_1)\frac{t(\alpha_1+iq_1)}{\alpha_2+iq_2-(\alpha_1+iq_1)^2}\widetilde{\omega}(\alpha-\mathbf{q}_{\Re};\mathbf{q}_1).$$

Recalling (37) and inverting (6) we derive for $\zeta M\zeta^{\dagger}$ in the x-space

$$(\zeta M \zeta^{\dagger})(x,x';q) = \frac{1}{(2\pi)^2}\int d\alpha \frac{e^{-i\alpha(x-x')}t(\alpha_1+iq_1)}{\alpha_2+iq_2-(\alpha_1+iq_1)^2}\widetilde{\chi}(x;\alpha_1+iq_1)\widetilde{\xi}(x';\alpha_1+iq_1),$$

and, finally, integrating over α_2, and summing up M_{Δ} as indicated in (141) we get for the hat version of the resolvent $\widehat{M}$

$$\begin{aligned}\widetilde{\widehat{M}}(x,x';q) &= \operatorname{sgn}(x_2-x_2') \\ &\times \frac{1}{2\pi i}\int d\alpha_1 \theta\left((q_2-2\alpha_1q_1)(x_2-x_2')\right)t(\alpha_1+iq_1)\widetilde{\Phi}(x;\alpha_1+iq_1)\widetilde{\Psi}(x';\alpha_1+iq_1) \\ &- \operatorname{sgn} q_1 \operatorname{sgn}(x_2-x_2')\sum_{n=1}^{N}\vartheta_n(q_1)\theta((q_2-2\lambda_{n\Re}q_1)(x_2-x_2'))\widetilde{\Phi}_n(x,q_1)\widetilde{\Psi}_n(x',q_1). \end{aligned} \quad (143)$$

It is worthwhile to note that in the r.h.s. of this formula for $\widetilde{M}$ the poles of $t(\mathbf{q}_1)$ at $\mathbf{q}_1=\lambda_n$ and at $\mathbf{q}_1=\overline{\lambda}_n$ in the first term generate discontinuities at $q_1 = \pm\lambda_{n\Im}$ which cancel exactly the discontinuities of the second term, due to the fact that the auxiliary Jost solutions $\Phi_n(q_1)$, $\Psi_n(q_1)$ are identically zero outside the strip $|q_1| \leq |\lambda_{n\Im}|$.

By using the reduction procedures indicated in (32) and (33) from this explicit expression for the resolvent one can derive the generalization of the standard Green's function for Jost and advanced/rearted solutions to the case of N solitons superimposed to a generic background. Thanks to the fact that they are derived from the resolvent they result both to be bilinear in terms of the Jost and auxiliary Jost solutions. The use of these Green's functions and other ones suggested by studying the properties of the resolvent is crucial in extending the IST to potentials obtained by perturbing the potential considered in this paper by adding to it an arbitrary smooth rapidly decaying function of both spatial variables. This is performed in a forthcoming paper following the method developed in [18].

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